MATB41 Week 10 Notes

1. Integral of Functions With 1 Var:

- Recall:

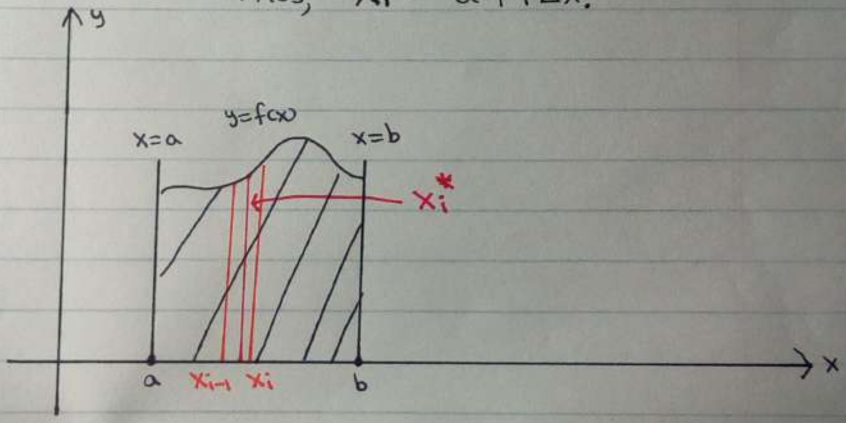
  Let $f: [a,b] \to \mathbb{R}$, $a<b$, $a,b \in \mathbb{R}$

  Choose an int $n>0$.

  Divide the interval $[a,b]$ into n equal subintervals.

  Note: $[a,b]$ has length $b-a$, so each of the subintervals has length $\frac{b-a}{n}$. $\Delta x = \frac{b-a}{n}$

  Thus, $x_i = a + i\Delta x$.



- Choose a sample point, $x_i^* \in [x_{i-1}, x_i]$ and form the Riemann Sum

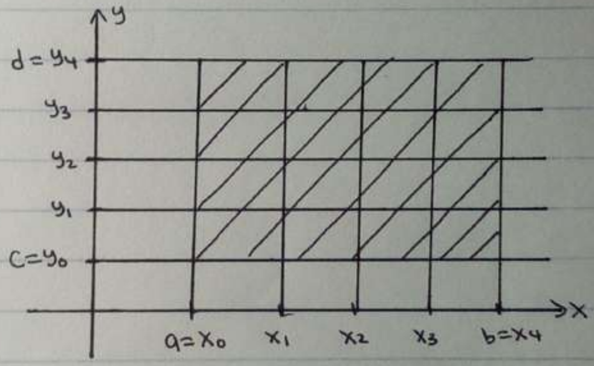
$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

- Note that $f(x_i^*)\Delta x$ is the area of the rectangle with base $\underbrace{[x_{i-1}, x_i]}_{\Delta x}$ and height $f(x_i^*)$.

- The integral on f on $[a,b]$ is
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

- If $f \geq 0$, $\int_a^b f(x)\,dx$ is the area of the region above $[a,b]$ under the graph of f.

2. Integrals of Functions with Multi-Variables:



- Let $f: [a,b] \times [c,d] \to \mathbb{R}$

- Choose ints $m, n > 0$.

- Divide the interval $[a,b]$ into m equal subintervals $[x_{i-1}, x_i]$. Note that $[a,b]$ has length $b-a$, so each of the subintervals has length $\Delta x = \frac{b-a}{m}$.

- Divide the interval $[c,d]$ into n equal subintervals $[y_{j-1}, y_j]$. Note that $[c,d]$ has length $d-c$, so each of the subintervals has length $\Delta y = \frac{d-c}{n}$.

- The rectangle $R = [a,b] \times [c,d]$ becomes $m \times n$ sub-rectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j] = R_{ij}$.

- Denote $\Delta A = \Delta x \Delta y = \frac{b-a}{m} \cdot \frac{d-c}{n}$

  which is the area of the sub-rectangle R_{ij}.

- Choose a sample point $(x_i^*, y_j^*) \in R_{ij}$. Then, $f(x_i^*, y_j^*) \Delta A$ is the vol of the small solid with base $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and height $f(x_i^*, y_j^*)$.

- $\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*) \Delta A$ approximates the volume of the solid lying under the graph of f and above the rectangle $R = [a,b] \times [c,d]$.

- $\iint_R f(x,y)\, dA = \lim_{n \to \infty} \lim_{m \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*) \Delta A$

  whenever the limit exists is called the **Double Integral** over R.

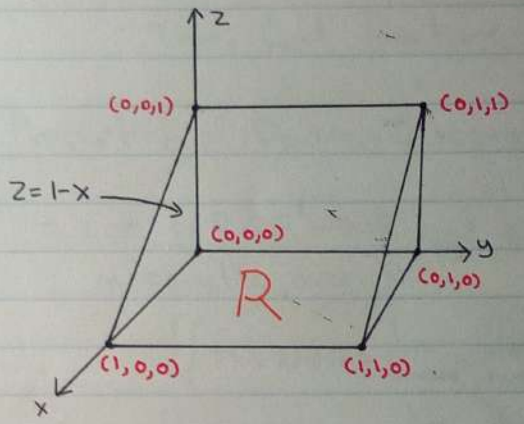
  - If $f \geq 0$, then $\iint_R f(x,y)\, dA$ is the vol of the solid lying under the graph of f above R.

  - MidPoint Rule:
    If we choose the sample points to be the center of the sub-rectangle R_{ij}, that is, $\overline{x_i}^*$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_j}^*$ is the midpoint of $[y_{j-1}, y_j]$, then we have the midpoint rule:

$$\iint_R f(x,y)\, dA = \sum_{j=1}^{n} \sum_{i=1}^{m} f(\overline{x_i}^*, \overline{y_j}^*) \Delta A$$

- The average value of f, denoted by f_{AVE}, is $\dfrac{\iint_R f(x,y)\,dA}{A}$ where A is the area of the rectangle R.

- E.g. Let $f(x,y) = 1-x$ over $R = [0,1] \times [0,1]$.



$$V = \iint_R (1-x)\,dA$$

$= \frac{1}{2}$ (Because this is half of a cube with length $=1$).

$A = 1 \times 1 = 1 \leftarrow$ The area of R.

$$f_{AVE} = \frac{\iint_R (1-x)\,dA}{A}$$

$$= \frac{1}{2}$$

- Thm: Let $f: R \subset \mathbb{R}^2 \to \mathbb{R}$ be a bounded real-valued function over the rectangle R, and suppose that the set of points where f is discont lies on a finite union of graphs of cont functions.
  Then, f is integrable over R.

- Thm: Cont functions are integrable.

3. Properties of Double Integrals:

- Let $f: R \subset \mathbb{R}^2 \to \mathbb{R}$ and $g: R \subset \mathbb{R}^2 \to \mathbb{R}$ be integrable over R.

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

$$\iint_R g(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} g(x_i^*, y_j^*)\,\Delta A$$

1. Homogeneity:
   If c is a constant in $\mathbb{R}$, $\iint_R c f(x,y)\,dA = c \iint_R f(x,y)\,dA$

Proof:

$$LS = \iint_R c f(x,y)\,dA$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} c f(x_i^*, y_j^*)\,\Delta A$$

$$= c \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

$$= c\iint_R f(x,y)\,dA$$

$$= RS$$

2. Linearity:

$$\iint_R f(x,y) \pm g(x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

Proof:

$$LS = \iint_R f(x,y) \pm g(x,y)\,dA$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \left(\sum_{j=1}^{n} \sum_{i=1}^{m} \left(f(x_i^*, y_j^*)\Delta A \pm g(x_i^*, y_j^*)\Delta A \right) \right)$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\Delta A \pm$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} g(x_i^*, y_j^*)\Delta A$$

$$= \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

$$= RS$$

3. Monotonecity:

If $f(x,y) \geq g(x,y)$ on R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

Proof:

If $m \le f(x,y) \le M$ on R, then

$$m \iint_R dA \le \iint_R f(x,y)\,dA \le M \iint_R dA$$

$$\iint_R m\,dA \le \iint_R f(x,y)\,dA \le \iint_R M\,dA$$

4. Additivity:

If R_i, $i = 1, 2, \dots, m$ are non-overlapping rectangles with

$$\iint_{R_i} f(x,y)\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta A \text{ and}$$

$$R = \bigcup_{i=1}^{m} R_i \text{, then } \iint_R f(x,y)\,dA = \sum_{i=1}^{m} \iint_{R_i} f(x,y)\,dA$$

5.

$$\left| \iint_R f(x,y)\,dA \right| \le \iint_R |f(x,y)|\,dA$$

4. Partial and Iterated Integral:

- Let $f: [a,b] \times [c,d] \to \mathbb{R}$
  If we fix y and let x vary from a to b, we can integrate $f(x,y)$ on the interval $[a,b]$ with respect to x.
  $\int_a^b f(x,y)\,dx$ is called the partial integration with respect to x.
  The result is the cross-sectional area that depends on y. This means that $\int_a^b f(x,y)\,dx$ is a function of y, denoted by $A(y)$.

- We may integrate $A(y)$ from c to d to obtain the volume of the solid.

$$V = \int_c^d A(y)\,dy$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

This is called an iterated integral.

- If $f \geq 0$ then $\iint_R f(x,y)\,dA$ is the volume of the solid lying under the graph of f above R.
  Therefore, $\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$

- In a similar way, we can define the partial integration of $f(x,y)$ with respect to y. We fix x and let y vary from $[c,d]$. $B(x) = \int_c^d f(x,y)\,dy$.
  Then, we integrate $B(x)$ from a to b to obtain $\int_a^b B(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$
  $$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

  This means that $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$

- Fubini's Thm:
  Let f be cont on the rectangular region $R = [a,b] \times [c,d]$. Then, the double integral of f over R may be evaluated by either of the two iterated integrals.

  I.e. $\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

- E.g. Evaluate the integral

$$\iint_R xy \, dA \text{ on } R = [1,2] \times [1,2]$$

Soln:

$$\iint_R xy \, dA = \int_1^2 \int_1^2 xy \, dx\,dy$$

$$= \int_1^2 y \int_1^2 x \, dx\,dy$$

$$= \int_1^2 y \left(\frac{x^2}{2} \Big|_1^2 \right) dy$$

$$= \frac{3}{2} \int_1^2 y \, dy$$

$$= \frac{3}{2} \left(\frac{y^2}{2} \Big|_1^2 \right)$$

$$= \frac{3}{2} \left(\frac{3}{2} \right)$$

$$= \frac{9}{4}$$

Alternatively:

$$\int_1^2\int_1^2 xy \, dy\,dx$$

$$= \int_1^2 x\int_1^2 y \, dy\,dx$$

$$= \int_1^2 x\left[\frac{y^2}{2}\Big|_1^2\right] dx$$

$$= \frac{3}{2}\int_1^2 x\,dx$$

$$= \frac{3}{2}\left(\frac{3}{2}\right)$$

$$= \frac{9}{4}$$

$$= \int_1^2\int_1^2 xy\,dx\,dy$$

- E.g. Let $f(x,y) = 1-x$ over $R = [0,1]\times[0,1]$
  This is a previous question. (Page 5)

$$\iint_R (1-x)\,dA = \int_0^1\int_0^1 (1-x)\,dx\,dy$$

$$= \int_0^1\int_0^1 1\,dx\,dy - \int_0^1\int_0^1 x\,dx\,dy$$

$$= \int_0^1 \left[x\Big|_0^1\right]dy - \int_0^1 \left[\frac{x^2}{2}\Big|_0^1\right]dy$$

$$= \int_0^1 1\,dy - \int_0^1 \frac{1}{2}\,dy$$

$$= \left[y\Big|_0^1\right] - \frac{1}{2}\left[y\Big|_0^1\right]$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

This is the same answer that we got in the question on page 5.

5. Choosing The Easier Iterated Integral:

- We know that $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

but sometimes, one of the iterated integral is much easier to work on and saves more time.

- E.g. Evaluate $\iint_R ye^{xy}\,dA$ where $R = [0,1] \times [0, \ln 2]$

Soln:

1. $\iint_R ye^{xy}\,dA = \int_0^1 \int_0^{\ln 2} ye^{xy}\,dy\,dx$

$$\int_0^{\ln 2} ye^{xy}\,dy$$

Integration by parts → Let $u = y \rightarrow du = dy$

Let $dv = e^{xy} \Rightarrow v = \frac{e^{xy}}{x}$

$$\int_0^{\ln 2} ye^{xy}\,dy = uv\Big|_0^{\ln 2} - \int_0^{\ln 2} v\,du$$

$$= \left(\Big|_0^{\ln 2} \right) - \int_0^{\ln 2} \frac{e^{xy}}{x}\,dy$$

$\frac{ye^{xy}}{x}$

$$= \frac{1}{x}\left[ye^{xy}\Big|_0^{\ln 2} - \int_0^{\ln 2} e^{xy}\,dy \right]$$

$$= \frac{1}{x}\left[(\ln 2)(e^{x\ln 2}) - \frac{e^{xy}}{x}\Big|_0^{\ln 2} \right]$$

$$= \frac{1}{x}\left[(\ln 2)(e^{x\ln 2}) - \left(\frac{e^{x\ln 2}}{x} - \frac{1}{x}\right)\right]$$

$$= \frac{(\ln 2)(e^{x\ln 2})}{x} + \frac{-e^{x\ln 2}+1}{x^2}$$

$$\int_0^1 \frac{(\ln 2)(e^{x\ln 2})}{x} + \frac{-e^{x\ln 2}+1}{x^2}\,dx \leftarrow \text{Type 2 improper integral}$$

$$= \lim_{A\to 0}\int_A^1 \frac{(\ln 2)(e^{x\ln 2})}{x} + \frac{-e^{x\ln 2}+1}{x^2}\,dx$$

$$= \lim_{A\to 0}\int_A^1 \frac{(\ln 2)(e^{x\ln 2})}{x}\,dx + \lim_{A\to 0}\int_A^1 \frac{1-e^{x\ln 2}}{x^2}\,dx$$

$$= (\ln 2)\lim_{A\to 0}\int_A^1 \frac{e^{x\ln 2}}{x}\,dx + \lim_{A\to 0}\int_A^1 \frac{1}{x^2}\,dx - \lim_{A\to 0}\int_A^1 \frac{e^{x\ln 2}}{x^2}\,dx$$

$$= e^{\ln 2} - 1 - \lim_{A\to 0}\left(\frac{e^{A\ln 2}-1}{A}\right)$$

$$= 2 - 1 - \lim_{A\to 0}\left(\frac{e^{A\ln 2}-1}{A}\right) \leftarrow \text{Form of } \tfrac{0}{0}.\ \text{L'hopital}$$

$$= 1 - \lim_{A\to 0}\left(\frac{\ln 2\,(e^{A\ln 2})}{1}\right)$$

$$= 1 - \ln 2$$

2. $\iint_R ye^{xy}\,dA = \int_0^{\ln 2}\int_0^1 ye^{xy}\,dx\,dy$

$= \int_0^{\ln 2} y \int_0^1 e^{xy}\,dx\,dy$

$= \int_0^{\ln 2} y \left[\frac{e^{xy}}{y} \Big|_0^1 \right] dy$

$= \int_0^{\ln 2} e^{y} - 1\,dy$

$= \int_0^{\ln 2} e^{y}\,dy - \int_0^{\ln 2} 1\,dy$

$= \left[e^{y} \Big|_0^{\ln 2} \right] - \left[y \Big|_0^{\ln 2} \right]$

$= e^{\ln 2} - e^{0} - \ln 2$

$= 2 - 1 - \ln(2)$

$= 1 - \ln(2)$

Note: Although both ways get us the same answer, method1 is more tedious and time consuming than method 2.

6. Double Integrals Over General Regions:

- Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ be a cont function and choose a rectangle R that contains the region D.

  Define $f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \text{ and } (x,y) \in R \end{cases}$

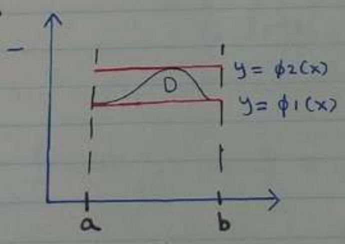
- The integral of f over the region D is given by: $\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA$

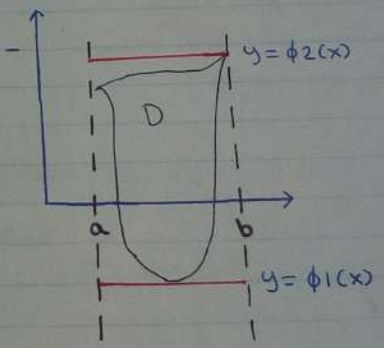
- Let ϕ_1 and ϕ_2 be two cont real-valued functions $\phi_i : [a,b] \to \mathbb{R}$, $i = 1, 2$ that satisfy $\phi_1 \leq \phi_2$ for all $x \in [a,b]$.

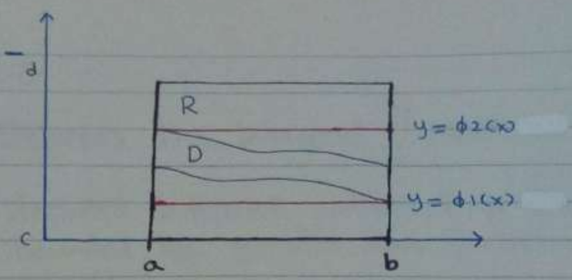
  $D = \left\{ (x,y) \,\middle|\, x \in [a,b] \text{ and } \phi_1(x) \leq y \leq \phi_2(x) \right\}$

  This is called y-simple.

- E.g.







$$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA$$

$$= \int_a^b \left(\int_c^{\phi_1(x)} f^*(x,y)\,dy + \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy + \int_{\phi_2(x)}^{d} f^*(x,y)\,dy \right) dx$$

$$= \int_a^b \left(0 + \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy + 0 \right) dx$$

Because $f^*(x,y)=0$ if $(x,y) \notin D$ and $f^*(x,y) \in R$.

$$= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

- Thm:

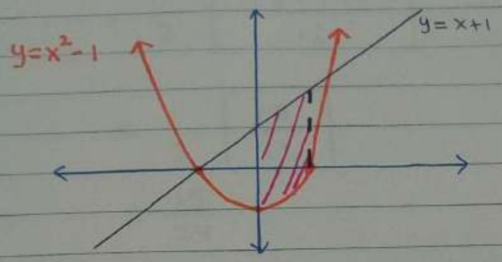
  Let $f(x,y)$ be cont on a y-simple region of D. Then:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

- E.g.

Express the integral $\iint_D xy\,dA$ as an iterated integral where D is the region bounded by $y=x^2-1$ and $y=x+1$ where $0 \le x \le 1$.

Soln:



$x^2-1 \le y \le x+1$

$0 \le x \le 1$

$D = (x,y)$ if $x^2-1 \le y \le x+1,\ 0 \le x \le 1$

$$\iint_D xy\,dA = \int_0^1 \int_{x^2-1}^{x+1} xy\,dy\,dx$$

$$= \int_0^1 x \int_{x^2-1}^{x+1} y\,dy\,dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \Big|_{x^2-1}^{x+1} \right] dx$$

$$= \frac{1}{2} \int_0^1 x\Big((x^2+2x+1) - (x^4-2x^2+1) \Big)\,dx$$

$$= \frac{1}{2}\int_0^1 -x^5 + 3x^3 + 2x^2 \, dx$$

$$= \frac{1}{2}\left[\left(-\frac{x^6}{6}\Big|_0^1\right) + 3\left(\frac{x^4}{4}\Big|_0^1\right) + 2\left(\frac{x^3}{3}\Big|_0^1\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{6} + \frac{3}{4} + \frac{2}{3}\right]$$

$$= \frac{1}{2}\left[\frac{-2+9+8}{12}\right]$$

$$= \frac{1}{2}\left[\frac{15}{12}\right]$$

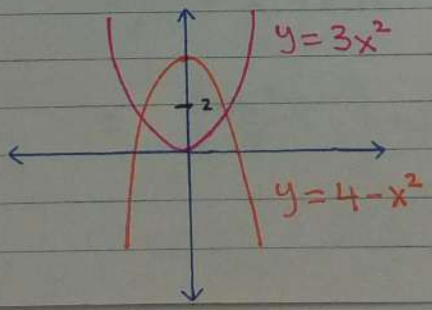
$$= \frac{15}{24}$$

$$= \frac{5}{8}$$

- E.g.

Express the integral $\iint_D (x^2+y)\, dA$ as an iterated integral where D is the region bounded by $y = 3x^2$ and $y = 4 - x^2$. Then, evaluate the integral.

Soln:



$y = 3x^2$ and $y = 4 - x^2$

$$3x^2 = 4 - x^2$$
$$4x^2 = 4$$
$$x^2 = 1$$
$$x = \pm 1$$

$$D = \{(x,y) \mid 3x^2 \le y \le 4 - x^2,\ -1 \le x \le 1\}$$

$$\iint_D (x^2+y)\,dA = \int_{-1}^{1}\int_{3x^2}^{4-x^2} x^2 + y \,dy\,dx$$

$$= \int_{-1}^{1} \left[x^2 y + \frac{y^2}{2} \Big|_{3x^2}^{4-x^2} \right] dx$$

$$= \int_{-1}^{1} x^2(4 - x^2 - 3x^2) + \frac{1}{2}\left((4-x^2)^2 - (3x^2)^2\right) dx$$

$$= \int_{-1}^{1} x^2(4 - 4x^2) + \frac{1}{2}(16 - 8x^2 + x^4 - 9x^4)\,dx$$

$$= \int_{-1}^{1} 4x^2 - 4x^4 + \frac{1}{2}(16 - 8x^2 - 8x^4)\,dx$$

$$= \int_{-1}^{1} 4x^2 - 4x^4 + 8 - 4x^2 - 4x^4 \,dx$$

$$= \int_{-1}^{1} -8x^4 + 8\,dx$$

$$= \left[\frac{-8x^5}{5} \Big|_{-1}^{1} \right] + 8\left[x \Big|_{-1}^{1} \right]$$

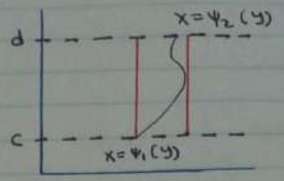
$$= \frac{-8}{5}(2) + 8(2) = 16 - \frac{16}{5} = \frac{80-16}{5}$$

$$= \frac{64}{5}$$

- Similarly, let ψ_1 and ψ_2 be two cont real-valued functions $\psi_i : [c,d] \to \mathbb{R}$, $i = 1, 2$ that satisfy $\psi_1 \le \psi_2$ for all $y \in [c,d]$.

  $D = \{(x,y) \mid y \in [c,d] \text{ and } \psi_1(y) \le x \le \psi_2(y)\}$

  This is called x-simple.

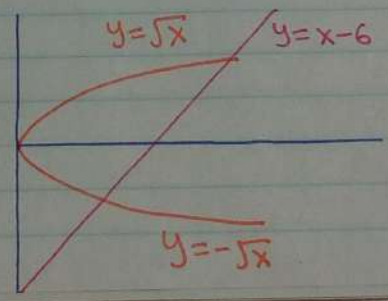


- Thm:
  Let $f(x,y)$ be cont on an x-simple region of D. Then:

  $$\iint_D f(x,y)\, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy$$

- E.g.
  Express the integral $\iint_D 4y^3 \, dA$ as an iterated integral where D is the region bounded by $x = y^2$ and $y = x - 6$.
  Then, evaluate the integral.

Soln:



$x = y^2$
$x = y+6$
$y^2 = y+6$
$0 = y^2 - y - 6$
$= (y-3)(y+2)$
$y = 3$ or $y = -2$
$-2 \le y \le 3$
$y^2 \le x \le y+6$

$$\iint_D 4y^3 \, dA = \int_{-2}^{3} \int_{y^2}^{y+6} 4y^3 \, dx \, dy$$

$$= \int_{-2}^{3} 4y^3 \int_{y^2}^{y+6} 1 \, dx \, dy$$

$$= \int_{-2}^{3} 4y^3 (y+6-y^2) \, dy$$

$$= \int_{-2}^{3} 4y^4 + 24y^3 - 4y^5 \, dy$$

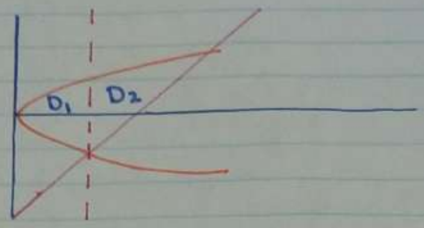
$$= \left[\frac{4}{5} y^5 \Big|_{-2}^{3} + 6y^4 \Big|_{-2}^{3} - \frac{2}{3} y^6 \Big|_{-2}^{3} \right]$$

$$= \frac{500}{3}$$

y- Simple:

$$\iint_D 4y^3\,dA = \iint_{D_1} 4y^3\,dA + \iint_{D_2} 4y^3\,dA$$

We know that the points of intersection of $x=y^2$ and $y=x-6$ are
1. $(4,-2)$
2. $(9,3)$



$$\iint_{D_1} 4y^3\,dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} 4y^3\,dy\,dx$$

$$\iint_{D_2} 4y^3\,dA = \int_4^9 \int_{x-6}^{\sqrt{x}} 4y^3\,dy\,dx$$

$$\iint_D 4y^3\,dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} 4y^3\,dy\,dx + \int_4^9 \int_{x-6}^{\sqrt{x}} 4y^3\,dy\,dx$$

$$= \frac{500}{3}$$

- Thm: Double integrals over regions can be represented as iterated integrals in 2 ways if D is both x-simple and y-simple. Sometimes, one way is easier to solve than the other.